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ASYMPTOTIC EXPANSION OF CERTAIN NUMBERS RELATED TO THE GAMMA FUNCTION

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Asymptotic expansion of certain numbers related to the gamma function

by

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ABSTRACT

An asymptotic expansion is derived for certain numbers which occur in the asymptotic expansion of the gamma function. This is done by application of the method of steepest descent to an integral representation and by making an essential use of the fact that the integrand is defined on a Riemann surface. The result is not new and the emphasis is on the way it is obtained.

KEY WORDS & PHRASES: *asymptotic expansion*

1. INTRODUCTION

The derivation of the asymptotic expansion for the gamma function belongs to the standard examples of asymptotic methods and it can be found in every textbook on this subject. The expansion can also serve as a tool for numerical evaluation and this makes the coefficients of some importance. There is no simple formula for the general coefficient, c_k say, but the first twenty are calculated and tabulated by WRENCH [4] (see also [2]).

In this note we will derive an asymptotic expansion as $k \to \infty$ for numbers d_k which are related to c_k by $c_k = \frac{(2k+1)!}{2^k k!} d_{2k+1}$. The same result was already obtained by WATSON [3] in connection with another problem. However, our method is quite different from WATSON's. He uses the method of Darboux which consists of a detailed examination of the Laurent expansion of the generating function for $\{d_k\}$ near the singularities on the radius of convergence. Our approach is in fact a direct application of the method of steepest descent. As is always the case with this method, the only difficulty consists of the choice of an appropriate contour. In this problem the best possible contour has the special feature that it must leave the Riemann sheet in which lies the point that has to be encircled.

2. DEFINITION OF THE NUMBERS d_k

The asymptotic expansion of the gamma function $\Gamma(z)$, which is sometimes called Stirling's formula, is

$$\Gamma(z) \sim z^z e^{-z} \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \frac{c_k}{z^k} . \tag{2.1}$$

A derivation of this formula can be found in [1, section 25]. There it is shown that the general coefficient c_k can be written as

$$c_k = \frac{(2k+1)!}{2^k k!} d_{2k+1} , \tag{2.2}$$

where the numbers d_k are implicitly defined as the coefficients in the power series expansion of $z(w)$ near $w = 0$, where $z - \ln(1+z) = \frac{1}{2}w^2$, $z(0) = 0$

and z real and positive for w real and positive. It is interesting to note that the same coefficients appear in the asymptotic expansion of the entire function $\frac{1}{\Gamma(z)}$, when this expansion is derived from Hankel's integral; see [1, section 30]. Since we want to obtain an asymptotic expansion of the numbers d_k we have to study the function z(w) in some more detail.

Let the function g(z) be defined by

$$(2.3) \qquad g(z) = z - \ln(1+z).$$

This function is many-valued with a branch point at z = − 1. We cut the plane along the negative real axis from −∞ to −1, and we obtain a Riemann surface with an infinite number of sheets. In the following the sheet with

$$(2.4) \qquad (2n-1)\pi < \operatorname{Im} \ln(1+z) \le (2n+1)\pi,$$

will be called n-sheet, n = ...,−1,0,+1,... . The equation Re g(z) = 0 has the same solution in each sheet, namely

$$x - \tfrac{1}{2}\operatorname{Ln}((1+x)^2+y^2) = 0 \quad \text{or} \quad y^2 = e^{2x} - (1+x)^2,$$

where z = x + iy and Ln denotes the principal value of the logaritmic function. The equation Im g(z) = 0 can be written as

$$\operatorname{tg} y = \frac{y}{x+1} ,$$

with the additional condition that in the n-sheet

$$2n\pi < y \le (2n+1)\pi \qquad \text{for } n > 0,$$
$$-\pi < y \le \pi \qquad \text{for } n = 0,$$
$$(2n-1)\pi < y \le 2n\pi \qquad \text{for } n < 0.$$

Now a graphical argument (see figure 1) shows that g(z) has one simple zero, z_n say, in each sheet, except in the 0-sheet where z = 0 is a double zero.

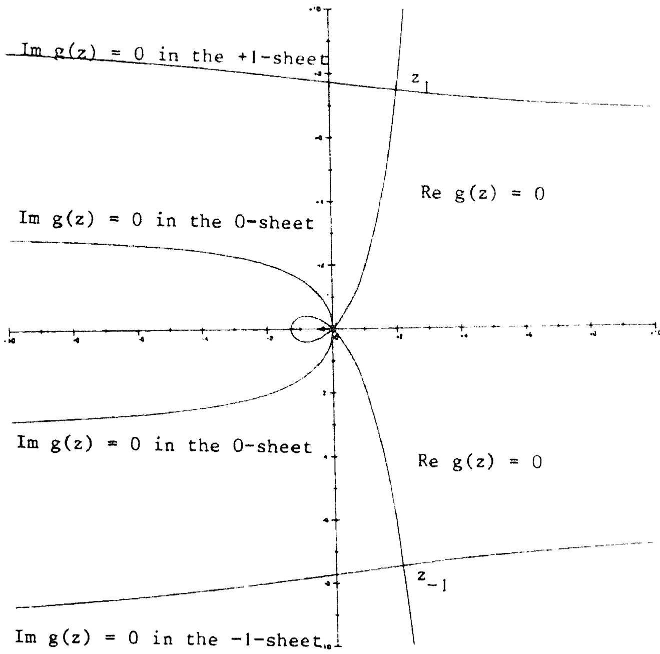

Figure 1

Let the function f(z) be defined by

(2.5) $$f(z) = \sqrt{2g(z)}.$$

Then f(z) can be made single-valued in the usual way. This means that we cut again the Riemann surface along each of the lines

$$\text{Im } g(z) = 0, \qquad \text{Re } g(z) \le 0.$$

Now the equation w = f(z) defines a mapping from the Riemann surface to the w-plane. The inverse function z(w) maps w = 0 onto z = 0 in the 0-sheet, it is a regular function near w = 0 and it takes the positive real axis in the w-plane to the positive real axis in the 0-sheet. Finally, the numbers d_k are defined by

(2.6) $$z(w) = \sum_{k=0}^{\infty} d_k w^k.$$

3. ASYMPTOTIC ANALYSIS

Our starting point for the asymptotic analysis of the coefficients d_k will be the contour integral representation which we deduce from Cauchy's integral formula

$$(3.1) \qquad kd_k = \frac{1}{2\pi i} \oint \frac{dz}{dw} w^{-k} \, dw.$$

We have chosen this particular representation with $\frac{dz}{dw}$ since we want to make a change of the integration variable from w to z. Under this transformation the closed contour around $w = 0$ is mapped onto a closed contour around $z = 0$ in the 0-sheet. So we have

$$(3.2) \qquad kd_k = \frac{1}{2\pi i} \oint \{f(z)\}^{-k} \, dz.$$

Let

$$(3.3) \qquad h(z) = \mathrm{Ln}\, g(z),$$

then the integrand of (3.2) can be written as $\exp(-\frac{k}{2} h(z))$. The saddle points are the solutions of

$$\frac{d}{dz} h(z) = \frac{1}{g(z)} \frac{z}{1+z} = 0.$$

These are precisely the points $z = 0$ in the sheets where $g(0) \neq 0$. The lines of steepest descent follow from

$$\mathrm{Im}\, h(z) = \mathrm{Im}\, h(0) \quad \text{or} \quad \arg g(z) = \arg g(0),$$

and since always $\mathrm{Re}\, g(0) = 0$ it follows that this is equivalent to $\mathrm{Re}\, g(z) = 0$. A local expansion yields that the line that passes through the valleys is tangent to $x = +y$ for $n < 0$ and to $x = -y$ for $n > 0$. Since $g(0) = 2n\pi i$, we need only to deform the path of integration in such a way that the saddle points $z = 0$ in the $\pm$ 1-sheets are passed (the others would not contribute significantly).

The appropriate contour is sketched in figure 2.

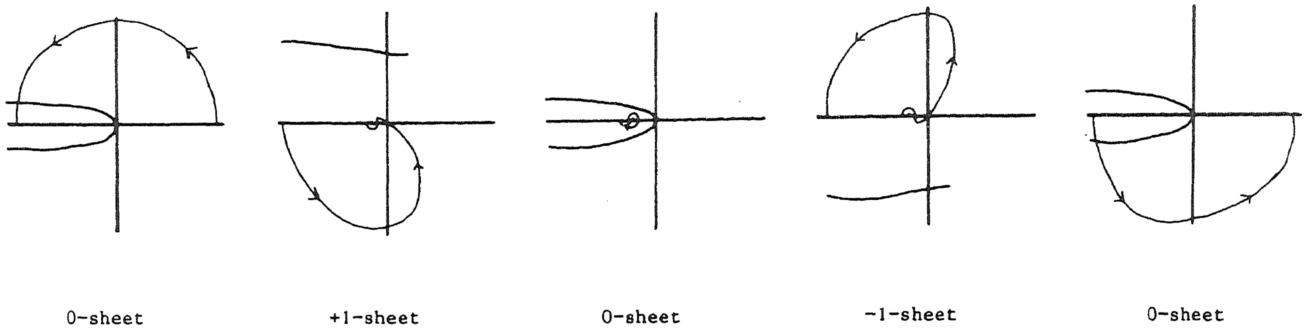

Figure 2

The contributions from the $\pm$ 1-sheets are complex conjugated. The final result is

(3.4) $$d_k = \left(\frac{-1}{2\sqrt{\pi}}\right)^k \frac{2\sqrt{2}}{k\sqrt{k}} \left\{\alpha_k + \left(\frac{3}{4}\alpha_k + \frac{\pi}{3}\beta_k\right)\frac{1}{k} + O\left(\frac{1}{k^2}\right)\right\}$$

where

(3.5) $$\alpha_k = -\sin\frac{k-1}{4}\pi, \qquad \beta_k = \cos\frac{k-1}{4}\pi.$$

From (3.4) it follows that the radius of convergence R of the series (2.6) is $2\sqrt{\pi}$. This fact can be deduced more directly. Since

$$\frac{dz}{dw} = \frac{w(1+z)}{z},$$

we know that the singularities of z(w) coincide with the zeros of z(w) with $w \neq 0$, i.e. with the inverse images of the points $z = 0$ in the sheets with $n \neq 0$. For $z = 0$ we have $w = \sqrt{-4n\pi i}$ and the result $R = 2\sqrt{\pi}$ follows at once. As a concluding remark we mention that the saddle points in the z Riemann surface are the images of the singularities of z(w) in the w-plane.

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